CRACK PROBLEMS IN PLANE AND ANTIPLANE ELASTICITY USING A SINGULAR INTEGRAL EQUATION

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A numerical integration formula for the investigation of the singular integral of Ioakimidis for classical crack problems in plane and antiplane elasticity is developed. The method is based on a modification of the Gauss-Chebyshev quadrature and the definition of finite part integral having an algebraic sigularity of $(-3/2)$ at the limits of integration. Once developed the procedure is applied to the determination of finite part integrals which have analytical solutions and the results are compared. Finally the integration for mula is applied to an actual crack problem and the stress intonsity factors are computed and presented.

Key Words: Singular Integral, Crack, Isotropic Polynomials, Stress Intensity, Quadrature

1. INTROCUCTION

The sigular integral equation of the first kind which arises in the investigation of straight cracks inside an isotropic elastic medium is

$$\frac{1}{\pi}\frac{d}{dx}\int_{-1}^{1}\frac{f(t)}{t-x}dt+\int_{-1}^{1}m(t,x)f(t)\,dt=-p(x),\quad -1<x<1 \tag{1}$$

In (1), $f(t)$ is an unknown function proportional to the crack opening displacement, $p(x)$ is the pressure distribution along the face of this crack, and $m(t,x)$ is a regular Kernel characteristic of the type of crack problem under investigation. Corresponding to (1) is the physical condition that the tips of the crack undergo no displacement

$$f(\pm 1)=0 \tag{2}$$

Performing the differentiation indicated in (1), one arrives at

$$\frac{1}{\pi}\int_{-1}^{1}\frac{f(t)}{(t-x)^2}dt+\int_{-1}^{1}m(t,x)f(t)\,dt=-p(x),\quad -1<x<1 \tag{3}$$

where the first integral denotes the finite part integral of Hadamard, that is, an integration technique in which fractional orders of infinty are removed(Kutt, 1975 and Hadamard, 1923). The singular integral equation (3) is transformed to

$$\frac{1}{\pi}\int_{-1}^{1}\frac{f'(t)}{t-x}dt+\int_{-1}^{1}l(t,x)f'(t)\,dt=p(x),\quad -1<x<1 \tag{4}$$

where

$$\frac{\partial}{\partial t}l(t,x)=m(t,x) \tag{5}$$

by performing an integration by parts on it.

The physical condition (2) is replaced by its equivalent

$$\int_{-1}^{1}f'(t)\,dt=0 \tag{6}$$

A final integration by parts is performed on (4) and one arrives at (7) which is Ioakimidis singular integral equation(Ioakimidis, 1983)

$$\frac{1}{\pi}\int_{-1}^{1}\ln|t-x|f''(t)\,dt+\int_{-1}^{1}K(t,x)f''(t)\,dt=-p(x),\quad -1\le x\le 1 \tag{7}$$

where

$$l(t,x)=\frac{\partial}{\partial t}K(t,x) \tag{8}$$

Performing an integration by parts on (6) one obtains

$$\int_{-1}^{1}(t-c)f''(t)\,dt=0 \tag{9}$$

where c is an arbitrary constant. The significance of (6) and (9) is that they make the determination of the constants of (5) and (8) unnecessary.

The unknown functions $f(t)$, $f'(t)$, and $f''(t)$ are replaced by regular unknown functions representing the proper singularities at ± 1.

$$f(t)=(1-t^2)^{1/2}g(t) \tag{10}$$

$$f'(t)=(1-t^2)^{-1/2}h(t) \tag{11}$$

$$f''(t)=(1-t^2)^{-3/2}q(t) \tag{12}$$

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For a description of how the general form of (10) is obtained, the reader is refered to(Erdogan et al., 1973). Equations (11) and (12) are obtained by takign successive derivative of (10). By performing these derivatives, one can see that $h(t)$ and $q(t)$ are

$$h(t)=-tg(t)+(1-t^2)g'(t) \tag{13}$$
$$q(t)=th(t)+(1-t^2)h'(t) \tag{14}$$

The stress intesity factors $K(\pm 1)$ at the crack tips are given by (Erdogan et al., 1973, Bucckner, 1973)

$$K(\pm 1)=g(\pm 1) \tag{15}$$
$$K(\pm 1)=\pm h(\pm 1) \tag{16}$$
$$K(\pm 1)=q(\pm 1) \tag{17}$$

The solution of singular integral equation introduced by Ioakimidis (Ioakimidis, 1983) can be summarized in the following procedure:

(1) Substituting (12) into (7) and (9) to arrive at

$$\frac{1}{\pi}\int_{-1}^{1}(1-t^2)^{-3/2}\ln|t-x|q(t)\,dt+\int_{-1}^{1}(1-t^2)^{-3/2} K(t,x)q(t)\,dt=-p(x),\quad -1\leq x\leq 1 \tag{18}$$
$$\int_{-1}^{1}(1-t^2)^{-3/2}(t-c)q(t)\,dt=0 \tag{19}$$

(2) Approximating $q(t)$ by finite polynomials $\phi_k(t)$ of the form

$$q(t)=q_n(t)=\sum_{k=0}^{n}a_k\phi_k(t) \tag{20}$$

(3) Determining the form of $\phi_k(t)$, by performing an integration by parts on the relation

$$\frac{1}{\pi}\int_{-1}^{1}(1-t^2)^{1/2}\frac{T_{k+1}(t)}{t-x}dt=U_k(x),\quad -1<x<1,\quad k\geq 0 \tag{21}$$

where $T_k(x)$ and $U_k(x)$ are Chebyshev polynomials of the first and second respectively, and it is arrived at

$$\frac{1}{\pi}\int_{-1}^{1}(1-t^2)^{-3/2}\ln|t-x|\phi_k(t)\,dt=-U_k(x),\quad -1<x<1,\quad k\geq 0 \tag{22}$$

In (22), $\phi_k(t)$ takes the form of one of the following equivalent expressions

$$\phi_k(t)=tT_{k+1}(t)+(k+1)(1-t^2)U_k(t) \tag{23}$$

or

$$\phi_k(t)=-\frac{k}{2}T_{k+2}(t)+\frac{k+2}{2}T_k(t) \tag{24}$$

(4) Integrating by parts the relationship

$$\int_{-1}^{1}(1-t^2)T_{k+1}\,dt=0,\quad k\geq 0 \tag{25}$$

one arrives at

$$\int_{-1}^{1}(1-t^2)^{-3/2}(t-c)\phi_k(t)\,dt=0,\quad k\geq 0 \tag{26}$$

The significance of (26) is that the condition of single value of displacement (19) is automatically satisfied when $\phi_k(t)$ is of the form of (23) or (24). Substituting (22) and (20) into (18), one obtains

$$\sum_{k=0}^{n}a_k[U_k(x)+\psi_k(x)]=P(x)+r_n(x) \tag{27}$$

where

$$\psi_k(x)=-\int_{-1}^{1}(1-t^2)^{-3/2}K(t,x)\phi_k(t)\,dt \tag{28}$$

and $r_n(x)$ is an error term due to the approximation of $q(t)$ by $q_n(t)$. To solve the system of $n+1$ linear equations represented by (27), Ioakimidis assured that $r_n(x_m)=0$, $m=0(1)n$ (29) at a set of collection points selected as the roots of the $T_{n+1}(x)$, Chebyshev polynomial and arrived at

$$\sum_{k=0}^{n}a_k[U_k(x_m)+\psi_k(x_m)]=P(x_m),\quad m=0(1)n \tag{29}$$

After determining the values of a_k, the stress intensity factors can be found from

$$K(1)=\sum_{k=0}^{n}a_k \tag{30}$$

or

$$K(-1)=\sum_{k=0}^{n}(-1)^k a_k \tag{31}$$

Equation (30) and (31) were arrived at by substituting (17) into (20) and using the relationships

$$\phi_k(1)=1 \tag{32}$$

and

$$\phi_k(-1)=(-1)^k \tag{33}$$

2. DEVELOPMENT OF THE QUADRATURE FORMULA

The bulk of the numerical work lies in the evaluation of the integral (28). In this section the Gauss-Chebyshev quadrature will be applied to the evaluation of this integral. Before this can be accomplished, (28) must be put into a suitable form by using Kutt's definition for a finite part integral having a singularity of the order $-3/2$(Kutt, 1975). The finite part integral definition that is of interest is

$$\int_{-1}^{0}\frac{f(t)}{(1+t)^{3/2}}=-2f(0)+2\int_{0}^{1}f'(t-1)\,t^{-1/2}dt \tag{34}$$

where $f(t)$ must satisfy the following conditions:

(a) $f(t)$ is continuous in the interval, $I\in[-1,\ 0]$.

(b) $f(t)$ is continuously differentiable once in a neighbourhood U of $t=-1\in I$.

To put (28) into the form of (34), the integration interval of

(28) is broken into two parts.

$$\phi_k(x) = -\int_{-1}^{0} \frac{(1-t)^{-3/2}\phi_k(t)K(t,x)}{(1+t)^{3/2}}dt + \int_{0}^{1} \frac{(1+t)^{-3/2}\phi_k(t)K(t,x)}{(1-t)^{3/2}}dt \quad (35)$$

where

$$(1-t^2)^{-3/2} = \frac{(1-t)^{-3/2}}{(1+t)^{3/2}} = \frac{(1+t)^{-3/2}}{(1-t)^{3/2}} \quad (36)$$

The second integral in (35) can be put into the form of (34) by using the transformation $t=-y$(Kutt, 1975) to arrive at

$$\int_{0}^{1} \frac{(1+t)^{-3/2}\phi_k(t)K(t,x)}{(1-t)^{3/2}}dt = \int_{-1}^{0} \frac{(1-y)^{-3/2}}{(1+y)^{3/2}}\phi_k (-y)K(-y,x)\,dy \quad (37)$$

Replacing the dummy variable y in (37) by t and substituting it into (35), the following equation is obtained

$$\phi_k(x) = -\int_{-1}^{0} \frac{(1-t)^{-3/2}}{(1+t)^{3/2}}[\phi_k(t)K(t,x) + \phi_k(-t)K(-t,x)]dt \quad (38)$$

In (38), the $f(t)$ corresponding to the definition (34) is

$$f(t) = (1-t)^{-3/2}[\phi_k(t)K(t,x) + \phi_k(-t)K(-t,x)] \quad (39)$$

Taking the derivative of (39), the following is obtained

$$f'(t) = [(K'(t,x)\phi_k(t) - K'(-t,x)\phi_k(-t)) + (K(t,x)\phi_k'(t) - K(-t,x)\phi_k'(-t))]/(1-t)^{3/2} + \frac{3}{2}[K(t,x)\phi_k(t) + K(-t,x)\phi_k(-t)] /(1-t)^{5/2} \quad (40)$$

where $K'(t,x) = 1(t,x)$. Using the expression

$$(1-t^2)U_k(t) = tT_{k+1}(t) - T_{k+2}(t) \quad (41)$$

and substituting (41) into (23), the expression for $\phi_k(t)$ becomes

$$\phi_k(t) = (k+2)\,tT_{k+1}(t) - (k+1)\,T_{k+2}(t) \quad (42)$$

Differentiating (42) and using the relationship

$$T_k'(t) = (k+1)\,U_k(t) \quad (43)$$

the following equation is obtained

$$\phi_k'(t) = (k+2)\,T_{k+1}(t) + (k+1)(k+2) [tU_k(t) - U_{k+1}(t)] \quad (44)$$

Using the expression

$$-T_{k+1}(t) = U_{k+1}(t) - tU_k(t) \quad (45)$$

the final expression for $\phi_k'(t)$ becomes

$$\phi_k'(t) = -k(k+2)\,T_{k+1}(t) \quad (46)$$

Applying the finite part definition (34) to (38), $\phi_k(x)$ becomes

$$\phi_k(x) = -[-4\phi_k(0)K(0,x) + 2\int_0^1 f'(t-1)\,t^{-1/2}dt] \quad (47)$$

where $f'(t-1)$ is obtained by replacing t by $(t-1)$ in (40). To evaluate the integral in (47), the following Gauss-Chebyshev quadrature will be used

$$\int_{-1}^{1} \frac{1}{\sqrt{1-z^2}}F(z)\,dt = \sum_{i=0}^{n} w_i F(z_i) \quad (48)$$

where

$$w_i = \frac{\pi}{(n+1)} \quad (49)$$

and

$$z_i = \cos\left(\frac{(2i+1)\pi}{(2n+2)}\right) \quad (50)$$

The integral in (47) will be transformed using the following transformation

$$t = \frac{(z+1)}{2} \quad (51)$$

and

$$dt = \frac{1}{2}dz \quad (52)$$

Performing the transformation, the integral becomes

$$\int_0^1 f'(t-1)\,t^{-1/2}dt = \frac{1}{2}\int_{-1}^{1}\left(z+\frac{1}{2}\right)^{-1/2} f'\left(\frac{z-1}{2}\right)dz \quad (53)$$

Multiplying (53) by

$$\frac{\sqrt{1-z^2}}{\sqrt{1-z^2}} = \frac{(1-z)^{1/2}(1+z)^{1/2}}{\sqrt{1-z^2}} \quad (54)$$

the weakly singular term (53) is eliminated and the integral becomes

$$\frac{\sqrt{2}}{2}\int_{-1}^{1} \frac{(1-z)^{1/2}}{\sqrt{1-z^2}} f'\left(\frac{z-1}{2}\right)dz \quad (55)$$

Using (48) to approximate (55) and substituting into (47), the expression for $\phi_k(x)$ becomes

$$\phi_k(x) = 4\phi(0)K(x,0) - \sqrt{2}\sum_{i=0}^{n} w_i(1-z_i)^{1/2} f'\left(\frac{z_i-1}{2}\right) \quad (56)$$

Using the relationships

$$T_{2k+1}(0) = 0 \quad (57)$$

and

$$T_{2k}(0) = (-1)^k \quad (58)$$

(56) takes the following forms

$$\phi_k(x) = 4(k+1)(-1)^{k/2} - \sqrt{2}\sum_{i=0}^{n} w_i(1-z_i)^{1/2} f'$$

$$\left(\frac{z_i-1}{2}\right) \quad (k:\text{even}) \tag{59}$$

$$\phi_k(x) = -\sqrt{2}\sum_{i=0}^{n} w_i (1-z_i)^{1/2} f'\left(\frac{z_i-1}{2}\right) \quad (k:\text{odd}) \tag{60}$$

where $f'\left(\frac{z_i-1}{2}\right)$, w_i, and z_i are given by (40), (49), and (50) respectively.

3. AN APPLICATION OF THE QUADRATURE FORMULA

To test quadrature formula, the integral which has the value given by the expression in (61) will be evaluated using the quadrature formula.

$$\int_{-1}^{1} \frac{x^q}{(1-x^2)^{3/2}} dx = \beta\left(-\frac{1}{2}, \frac{q-1}{2}\right), \quad q \text{ is an integer} \geq 0 \tag{61}$$

In (61) $\beta\left(-\frac{1}{2}, \frac{q-1}{2}\right)$ denotes the Beta function evaluated at the arguments $-\frac{1}{2}$, $\frac{q-1}{2}$. For the case $q>1$, applying the quadrature formula to (61) results in

$$\int_{-1}^{1} \frac{x^q}{(1-x^2)^{3/2}} dx = \frac{\sqrt{2}\pi}{(n+1)} \sum_{i=0}^{n} (1-x_i)^{1/2} \left[q(1-y_i)^{-3/2}[y_i^{q-1}-(-y_i)^{q-1}] + \frac{3}{2}(1-y_i)^{-5/2}[x^q+(-x)^q]\right] \tag{62}$$

where

$$x_i = \cos\left(\frac{(2i+1)\pi}{(2n+2)}\right) \tag{63}$$

and

$$y_i = \frac{(x_i-1)}{2} \tag{64}$$

In Table 1, when $n=10$ the numerical values obtained by evaluating the integral using (64) for even values of q up to 20 are presented along with the analytical values from (61).

Table 1 Evaluation of (64)

q	Gauss-Chebyshev	β
2	− 3.14154264	− 3.14159266
4	− 4.17238895	− 4.71238898
6	− 5.89048623	− 5.89048623
8	− 6.87223393	− 6.87223393
10	− 7.73126318	− 7.73126317
12	− 8.50438951	− 8.50438949
14	− 9.21308864	− 9.21308862
16	− 9.87116641	− 9.87116638
18	−10.4881143	−10.4881143
20	−11.0707873	−11.0707873

4. APPLICATION OF QUADRATURE FORMULA TO A CRACK PROBLEM

The crack problem under investigation is taken from (Erdogan et al., 1973)

In Fig. 1, the composite plane is loaded in such a way that the normal component of the crack surface loading is the only external surface load. The starting integral equation for the investigation of this problem is

$$\frac{1}{\pi}\int_a^b \frac{\phi(r_0)}{r_0-r} dr_0 + \frac{1}{\pi}\int_a^b H(r, r_0)\phi(r_0)\, dr_0 = \frac{1+k_1}{2\mu_1} P(r) \tag{65}$$

where μ_1 and μ_2 are the shear moduli, $K_i=(3-v_i)/(1+v_i)$ for generalized plane stress, $K_i=3-4v_i$ for plane strain, v_i is Poisson's ratio and m is defined as μ_2/μ_1. In (65) $H(r, r_0)$ has this form

$$H(r, r_0) = \frac{1}{2(1+mk_1)(m+k_2)} \left\{ \frac{1}{r+r_0}[(1+mk_1)(m+k_2) - m(1+K_1)(1+mk_1) - 3(1-m)(m+k_2)] + 12(1-m)(m+k_2)\frac{r}{(r_0+r)^2} - 8(1-m)(m+k_2)\frac{r^2}{(r_0+r)^3} \right\} \tag{66}$$

To normalize the interval (a,b), the following equations will be used

$$x = \frac{(r-c)}{a_0}, \quad t = \frac{(r_0-c)}{a_0}, \quad a_0 = \frac{(b-a)}{2} \tag{67}$$

$$\frac{1+k_1}{2\mu} P(r) = P(x), \quad \phi(r_0) = f'(t) \tag{68}$$

$$\frac{1}{\pi} H(r, r_0) = \frac{1}{a_0} 1(x, t) \tag{69}$$

Performing the normalization (65) becomes

$$\frac{1}{\pi}\int_{-1}^{1} \frac{f'(t)}{t-x} dt + \int_{-1}^{1} 1(x, t) f'(t)\, dt = P(x) \tag{70}$$

where

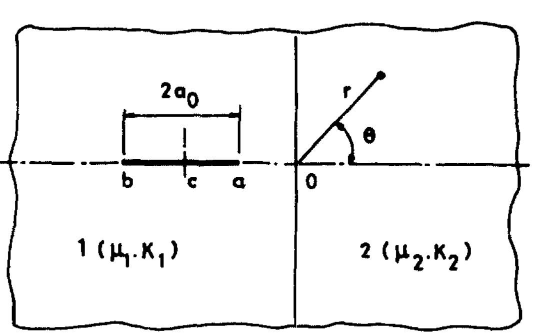

Fig. 1 A finite crack perpendicular to the bi-material interface

$$1(x,t)=\frac{1}{2\pi(1+mk_1)(m+k_2)}\left\{\frac{1}{x+B+2D}[(1+mk_1)(m+k_2)-m(1+k_1)(1+mk_1)-3(1-m)(m+k_2)]+12(1-m)(m+k_2)\frac{x+B}{(x+B+2D)^2}-\frac{8(1-m)(m+k_2)(x+B)^2}{(x+B+2D)^3}\right\} \quad (71)$$

and

$$D=\frac{c}{a_0} \quad (72)$$

Integrating (71) with respect to t, $K(t,x)$ in (7) is found to be

$$K(t,x)=\frac{1}{2\pi(1+mk_1)(m+k_2)}\left\{[(1+mk_1)(m+k_2)-m(1+k_1)(1+mk_1)-3(1-m)(m+k_2)]\ln(x+t+2D)+\frac{12(1-m)(m+k_2)(x+D)}{(x+t+2D)}+\frac{4(1-m)(m+k_2)(x+D)^2}{(x+t+2D)^2}\right\} \quad (73)$$

In the following tables, the stress intensity factors calculated from (29), (30), and (31) will be presented along with the values computed in (Erdogan et al., 1973).

Table 2 Stress intensity factors for case $m=0$ and $p(x)=-p_0$ a constant $n=10$, $Q=25$

D	Gauss-Chebyshev Quadrature		F. Erdogan et al.,(1973)	
	$\frac{K(b)}{p_0(a_0)^{1/2}}$	$\frac{K(a)}{p_0(a_0)^{1/2}}$	$\frac{K(b)}{p_0(a_0)^{1/2}}$	$\frac{K(a)}{p_0(a_0)^{1/2}}$
1.01	1.327	3.715	1.330	3.720
1.05	1.251	2.152	1.254	2.159
1.10	1.208	1.756	1.211	1.759
1.15	1.180	1.573	1.813	1.573
1.20	1.160	1.461	1.163	1.464
1.25	1.144	1.385	1.146	1.388
2.00	1.051	1.089	1.054	1.091
5.00	1.006	1.008	1.009	1.011
10.0	0.999	0.999	1.002	1.003

Table 3 Stress intensity factors for case $m=23.08$ and $p(x)=-p_0$ for plane strain $n=10$, $Q=25$

β	Gauss-Chebyshev Quadrature		F. Erdogan et al.,(1973)	
	$\frac{K(b)}{p_0(a_0)^{1/2}}$	$\frac{K(a)}{p_0(a_0)^{1/2}}$	$\frac{K(b)}{p_0(a_0)^{1/2}}$	$\frac{K(a)}{p_0(a_0)^{1/2}}$
1.10	0.8994	0.6677	0.8985	0.6674
1.15	0.9060	0.7182	0.9051	0.7178
1.25	0.9174	0.7843	0.9165	0.7838
2.00	0.9626	0.9357	0.9616	0.9349
5.00	0.9939	0.9923	0.9929	0.9912
10.0	0.9993	0.9990	0.9981	0.9979

Table 4 Stress intensity factors for case $m=23.08$ and $p(x)=-p_0x$ for plane strain $n=10$, $Q=25$

β	Gauss-Chebyshev Quadrature		F. Erdogan et al.,(1973)	
	$\frac{K(b)}{p_0(a_0)^{1/2}}$	$\frac{K(a)}{p_0(a_0)^{1/2}}$	$\frac{K(b)}{p_0(a_0)^{1/2}}$	$\frac{K(a)}{p_0(a_0)^{1/2}}$
1.10	0.5084	−0.4058	0.5084	−0.4058
1.15	0.5075	−0.4300	0.5075	−0.4300
1.25	0.5062	−0.4564	0.5062	−0.4564
2.00	0.5019	−0.4943	0.5019	−0.4942
5.00	0.5016	0.4998	0.5002	−0.4997
10.0	0.5000	0.5000	0.5000	0.4999

6. CONCLUSION

The advantage of the proposed singular integral equation is that its kernel is weakly singular and that the numerical solution of singular integral equations with logarithmic singularities is more classical than the numerical solution of Cauchy type singular integral equations. Also the physical condition is automatically satisfied because of the choice of $q(t)$. Finally, $f''(t)$ has a physical interpretation proportional to the second derivative of the crack opening displacement almost equal to the curvature of the deformed edges of a straight crack after moving away from the crack tip. The simplicity of the quadrature makes possible the evaluation of a large number of linear equations (m) in (29). Application of Gauss-Chebyshev Quadrature to a finite crack perpendicular to the bi-material interface showed good agreement with the numerical results by Erdogan (Erdogan et al., 1973).

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